Links Between Early Arithmetical Knowledge And Early Space And Measurement Knowledge: An Exploratory Study.

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This paper focuses on better understanding children's early space and measurement knowledge. Five Year 1 students from each of two schools and five Kindergarteners from one of those schools, were given an initial and final assessment in arithmetic, space and measurement. Students' advancements in arithmetic and space were not accompanied by advancements in measurement. Arithmetically more able students were found to be more able on space and measurement tasks and there was great variation in students' space and measurement knowledge.

Whilst research has provided useful models of student's early arithmetical knowledge and its development (see Wright, 1994; 1996) little research related to early geometric knowledge seems available. Historically the most influential research in this area has been the work of Pierre and Dina van Hiele (van Hiele, 1959; van Hiele, 1986) and Piaget and Inhelder (Gruber & Voneche, 1977). The van Hieles postulated five levels of cognitive development in geometric thinking: recognition, analysis, ordering, deduction and rigour. Piaget believed that children's organisation of geometric ideas follows a definite and logical rather than historical order. Initially the child constructs topological relations, such as closure, connectedness and continuity. Projective and Euclidean relations are developed later (Gruber & Voneche, 1977). Both the theories of Piaget and the van Hieles emphasise the role of students in actively constructing their own knowledge and stress the importance of challenging the learner's thinking. Both theories have been influential in the teaching of geometry.

During the last decade school mathematics has changed from a focus on procedures to an emphasis on patterns and relationships (Wheatley, 1990; National Statement, 1990), the construction of which is largely dependent on spatial sense (NCTM, 1989). Wheatley (1990) likens spatial sense to imagery and maintains that "spatial sense is indispensable in giving meaning to our mathematical experience" (p. 15). He deplores the mathematics program typical of most US elementary schools where the study of geometry is reduced to the identification and naming of simple geometric shapes and emphasises the importance of taking account of students' spatial sense, both in teaching and assessment (p.15).

In more recent times there has been an increased interest in research on students' spatial thinking, although most studies appear to have focussed on older groups of children or on very specific areas of children's spatial thinking. For instance, Wheatley and Cobb (1990) presented children with shape-covering tasks to determine the level of their understanding of area; Mitchelmore (eg 1994) focussed on the development of specific spatial concepts, such as angles and parallels; Pegg & Davey (1989) and Pegg(1992) seem largely concerned with descriptors of students' spatial understanding; and Leeson (1995) conducted two studies involving kindergarteners' spatial construction of three-dimensional shape. Leeson found that young children were capable of significant geometrical thinking and suggested that many children are under-challenged in geometry in their first year of school. Thorpe (1995) who views spatial relationships as having "wider parameters than the traditional subject of geometry" (p. 517) investigated pre-school students' learning and use of spatial concepts from an interactive perspective whilst she was an observer in the students' normal setting.

Owens who has worked extensively in the area of assessment of students' spatial concepts (eg 1992a; 1992b; 1994) studied a broad range of children's spatial understanding. She devised a group 'pencil and paper' test which assessed children's ability to recognise, reproduce and analyse shapes (1992). Owens maintained that "despite the emphasis on qualitative approaches to the research the relationship between spatial abilities and spatial concept development is not yet clear" (1992, p.2). Owens, Mitchelmore, Outhred and Pegg (1996) report a study by Brodie (1994) in which 138 teachers were surveyed in regard to the Space strand in Mathematics K-6. It was found that teachers' attitudes towards the strand were less positive than to the other strands; less time was spent on the strand; theoretical knowledge of students' spatial learning was poor; and their understanding of the scope of geometry was rooted in their own school learning.

There appears to be little research available on students' early measurement knowledge. As well, one could argue that much early measurement knowledge is not distinct from and might be subsumed under early arithmetical knowledge. The theory of the development of early arithmetical knowledge by Steffe and Cobb (1988) for example, includes notions such as units and units of units, which seem also relevant to the development of early measurement knowledge. Given that measurement is an important topic of early mathematics curricula, its inclusion in this study is warranted.

The present study was carried out in response to a perceived need to gain a deeper understanding of the spatial and measurement knowledge of students in their first or second years of formal schooling. An important aim is to extend to the assessment of students' early spatial and measurement knowledge, an interview-based approach to the assessment of early arithmetical knowledge. Additional aims of the project include the documentation of students' advancements in spatial and measurement knowledge over the course of a school year and a comparison of students' arithmetical, spatial and measurement knowledge. One question addressed by this study is: What spatial and measurement knowledge is possessed by students in Kindergarten or Year 1 in NSW? Research techniques developed in the Maths Recovery project (see Wright, in press) were used in this project. These include (a) student assessment based on individual interview in which the student is presented with tasks which are problematic for them; (b) routine videotaping of assessment interviews for subsequent analysis; and (c) an explicit focus on students' strategies and meaning as demonstrated in the course of the assessment interview.

Assessment Schedules

During the first half of 1996, assessment schedules in early space and measurement were developed. These schedules were informed by the national and NSW curriculum documents (A National Statement on Mathematics for Australian Schools, 1990; Mathematics K-6,1989), the work of one of the authors (Leeson) and current research into space and measurement. The two assessment schedules were trialled in April with ten students, five from each of two schools, and modifications were made. The final versions of these schedules each required approximately 45 minutes for completion. The space schedule comprised 20 activities and measurement 19.

The space schedule covered three broad areas of students' spatial knowledge: two dimensional (2D) space; three dimensional (3D) space and position. Within the area of 2D space, students' abilities to name, recognise, draw, construct, classify and identify were assessed. Students' understanding of symmetry and tiling with seven-piece tangrams also formed part of this assessment schedule. In the area of 3D space, students were required to sort, classify and name objects. They were also required to construct 3D shapes from a 2D representation, match pictorial representations of the nets of shapes with solids, determine the number of cubes in a structure from its pictorial representation, and provide explanations of their solutions. Simple grids were used to assess students' understanding of position. Students were required both to place an object in a specified position on the grid and to describe the position of objects placed onto the grid by the interviewer.

The measurement schedule involved the use of materials (such as straws, popsticks, cubes, boxes, various containers, balloons, and balls) and informal units. Children were required to estimate, measure, compare and order length, perimeter, area, capacity, volume, mass and time.

Method

Fifteen students from local schools, who were not part of the trial phase, participated in the study. There were five Kindergarten and five Year 1 students from one school (School A), and five Year 1 students from a second school (School B). Each student was given an initial assessment early in the third school term and a final assessment toward the end of the fourth (final) term. The assessments consisted of three interviews - arithmetic, space and measurement. The arithmetic interview was one which was developed and used in earlier research (eg Wright, 1994; Wright, Stanger, Cowper & Dyson, 1996). Analysis of the arithmetic interview includes determination of student's stage of early arithmetical learning, viz: Stage 0 - Emergent; Stage 1- Perceptual; Stage 2 - Figurative; Stage 3 - Initial Number Sequence; Stage 4 - Intermediate Number Sequence; or Stage 5 - Facile Number Sequence (see Wright, 1994, pp 26-28).

On each of the spatial and measurement schedules it was decided that comparisons could more readily be made if some numerical value could be assigned to each student's performance. Analysis of the space and measurement assessments involved writing protocols, ie descriptions of each student's solutions as well as student's and interviewer's statements. Subsequent analysis involved assigning a numerical value to the student's solution of each task. The value was determined according to task-specific criteria. For instance, Activity 3 in the space assessment was made up of three subtasks and involved students selecting, from a total of 19 shapes those which were 'like' one held up by the interviewer and then explaining the likeness. There were two circles, four squares, two hexagons, three rectangles and eight triangles to choose from. The interviewer first held up a square, then a rectangle, then a triangle. The maximum score available for this activity was six. One point was allocated for selecting all the correct shapes in each sub-task and one point was awarded for each appropriate explanation. For example, Jack was allocated five out of six for Activity 3. He correctly selected all the squares and rectangles and gave satisfactory explanations for his selections, but only selected three of the eight triangles. He explained that the three chosen triangles were like the shape held up by the interviewer because they "all have three sides and three corners". Jack was awarded four points for selecting all the squares and rectangles and offering appropriate explanations. He was given one point for his explanation and zero for his selection because he had not selected all the triangles, although his explanation should have encompassed all of them. When asked why the remaining five had not been chosen Jack replied that they "had really long sides and two short ones". A similar scoring system was employed in assigning numerical values to each students' solution of each of the measurement tasks.

In this manner each student was allocated a score for their performance on each of the space and measurement tasks. These scores were summed and this led to an aggregate score for each student's performance on the initial and final space and measurement assessments. The maximum possible score was 70 on the space assessment and 48 on the measurement assessment.

Results

Table 1 shows the total scores of the 14 students who were given the initial and final space assessments. The table also includes the grade level - Year 1(1) or Kindergarten (K)- and school of each student, together with each student's total score, expressed as a percentage of the maximum possible score of 70. Table 2 lists the total scores of the 13 students who completed the initial and final measurement assessments. This table also includes each student's school, grade level and score as a percentage of the maximum possible score of 48. Table 3 shows a comparison of students' early arithmetical, early space and early measurement knowledge The table lists each student's grade level, Stage (Wright, 1994) on initial and final interviews, and each student's total score in space and measurement for the initial and final interviews. In Table 3 students have been ordered according to their Stage in the initial interview.

Table 1
Initial and final scores on space assessments.

Name	School	Grade	Initial Assessment (max. 70)		Final Assessment (max. 70)	
Beth	A	1	66	(94.3)	68	(97.1)
Jack	A	1	62	(88.6)	68	(97.1)
Jessica	A	1	60	(85.7)	60	(85.7)
Jessamy	A	1	59	(84.3)	65	(92.8)
Keenan	B	1	55	(78.6)	59	(84.3)
Jason	B	1	54	(77.1)	59	(84.3)
Andrew	A	1	54	(77.1)	66	(94.3)
Isaac	A	K	43	(61.4)	43	(61.4)
Kellie-Anne	B	1	42	(60.0)	53	(75.7)
Hollie	A	K	39	(55.7)	50	(71.4)
Carmin	A	K	38	(54.3)	*	*
Chloe	A	K	37	(52.9)	39	(55.7)
Michelle	B	1	37	(52.9)	46	(65.7)
Mark	B	1	35	(50.0)	45	(64.3)
Dwyatt	A	K	35	(50.0)	46	(65.7)

Note: *Carmin left the school prior to the final interviews.
Scores are from a possible total of 70.

Table 2
Initial and final scores on measurement assessments.

Name	School	Grade	Initial Assessment (max. 48)		Final Assessment (max. 48)	
Jack	A	1	40	(83.3)	38	(79.2)
Andrew	A	1	39	(81.3)	39	(81.3)
Jason	B	1	37	(77.1)	37	(77.1)
Jessamy	A	1	36	(75.0)	39	(81.3)
Jessica	A	1	36	(75.0)	31	(64.6)
Hollie	A	K	33	(68.8)	30	(62.5)
Kellie-Anne	B	1	33	(68.8)	32	(66.7)
Carmin	A	K	32	(66.7)	n/a	n/a
Keenan	B	1	31	(64.6)	30	(62.5)
Mark	B	1	30	(62.5)	30	(62.5)
Isaac	A	K	29	(60.4)	30	(62.5)
Michelle	B	1	28	(58.3)	36	(75.0)
Dwyatt	A	K	24	(50.0)	28	(58.3)
Chloe	A	K	21	(43.7)	16	(33.3)
Beth	A	1	#	#	37	(77.1)

Note: # Beth was unavailable for the initial measurement assessment.
Scores are from a possible total of 48.

Table 3
Initial (A1) and final (A2) Stage and scores on space and measurement assessments.

Name	Grade	Number		Space		Measurement	
		A1	A2	A1	A2	A1	A2
Jessamy	1	4	5	59	65	36	39
Beth	1	3	5	66	68	n/a	37
Jack	1	3	4	62	68	40	38
Andrew	1	3	3	54	66	39	39
Jason	1	3	4	54	59	37	37
Jessica	1	3	3	60	60	36	31
Kelli-Anne	1	3	3	42	53	33	32
Michelle	1	2/3	3	37	46	28	36
Keenan	1	2	3	55	59	31	30
Mark	1	2	2	35	45	30	30
Carmin	K	1/2	*	38	*	32	n/a
Dwyatt	K	1/2	2	35	46	24	28
Isaac	K	1	1	43	43	29	30
Holllie	K	1	1	39	50	33	30
Chloe	K	0	1	37	39	21	16

Note: * Carmin left the school prior to the final interviews.
Students' scores have been ordered by their Stage as determined from their initial assessments.

Summary points from the results.

1. Great variation is evident in students' spatial knowledge, not only across school years but within school years as well. Thus, Dwyatt, a Kindergarten student gained a much lower score (35) on his initial space interview than Beth, a Year 1 student (66). Similarly, Mark (35), who was also in Year 1, scored low in comparison to Beth. This trend was also evident in students' scores on the measurement assessment. For instance, Jack, a Year 1 student, scored considerably higher than Chloe, a Kindergarten student, and Michelle who was in Year 1.

2. Most students (twelve of 14) showed advancements in spatial knowledge over the four month period. The two students who did not advance scored the same total in the initial and final space assessments.

3. Six of the 14 students showed improvements of more than 10% of the possible maximum in their final space assessment scores.

4. Only 4 of 13 students showed improvements in their overall score on measurement, and one of these improved by one point only. Three students attained the same score and six attained lower scores, two showing decreases of five points. Moreover, the increases shown by students were small and only one student improved by more than 10% of the possible maximum on her final interview.

5. Students' overall scores, when expressed as a percentage of the maximum possible score, were higher on both initial and final space assessments than their initial and final measurement scores. For instance scores on the initial space assessment ranged from 94.3% to 50%, whereas scores on the initial measurement assessment ranged from 83.3% to 43.7%.

6. Most students who were more able arithmetically showed advancements between all initial and final interviews

7. Generally speaking the students who were more able arithmetically were also more able on the space and measurement assessments. For example, Jessamy, who was assessed at Stage 4 from her initial number interview, showed a similar high ability with spatial and measurement tasks in her initial and final assessments.

8. The student (Chloe) who was assessed at Stage 0 scored low in her initial and final space assessments and attained the lowest scores of 13 students in her initial and final measurement assessments. Her gains in spatial knowledge were very low and she scored considerably lower in her final measurement assessment than in her initial measurement assessment.

Discussion

The lack of improvement in students' scores on measurement may have been due in part to the inclusion of estimation and conservation tasks in the measurement assessment. Five of the 19 tasks involved student estimations of length, perimeter or area, and three of the tasks involved students' abilities to conserve. Points were awarded for giving exact estimations, close estimations, and in some instances for just giving an estimate. Students were given one point if they were able to conserve or zero if they were not able to conserve. Some student's estimates varied from initial to final assessments, as did their ability to conserve. For instance, five students scored higher on Task 2 in the initial assessment than they did in the final assessment because their initial estimates were more accurate. Similarly, only two of 13 students conserved consistently on Task 13 in the initial and final interviews. It is conceivable that students' ability to conserve may not be firmly established at this age. Similarly students' estimation skills, particularly in relation to perimeter and area, may still be in a formative phase for these age groups.

The apparent link between high-scoring students' numerical ability and their spatial and measurement ability warrants further investigation. Similarly, the case of Chloe raises some interesting questions. Is there a tendency for children who are low-achievers in early arithmetic to show correspondingly low levels in space and measurement? Thus there may be a case for extending maths recovery kinds of programs to encompass all areas of mathematics and not just arithmetic.

Though not dealt with in this paper, of interest to the authors are the kinds of strategies students used in task solution. Several different student strategies were noted and it appears that a larger study may result in the identification of levels in children's thinking in the areas of space and measurement.

Following this exploratory study work needs to be done in refining the assessment schedules for early space and measurement. For instance, the value of a few tasks which were easily accomplished by all students is questionable. Tasks need to be designed so that they are both challenging and discriminatory.
Other factors warranting consideration are possible gender differences and the effects of different schools on the participants.

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